Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 273 (2004) 373–383

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Boundary valuation domains

Jack Maney

Department of Mathematics, North Dakota State University, Fargo, ND 58105-5075, USA

Received 13 December 2002

Abstract

A *half factorial domain* (HFD) R is an atomic domain where, for any collection of irreducibles $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n\}$, with $\alpha_1\alpha_2\cdots\alpha_m = \beta_1\beta_2\cdots\beta_n$ we have $n = m$. In a paper by J. Coykendall [Comm. Algebra 27 (1999) 3153–3159], a generalization of the length function of Zaks [Israel J. Math. 37 (1980) 281–302], called the *boundary map*, was introduced. A new class of HFD's—called boundary valuation domains—are defined and studied using this map.

1. Introduction and background

In this paper, all rings are assumed to be commutative with identity. By $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, we mean the natural numbers, the integers, and the rational numbers (respectively). If S is a subset of a ring R, then by S^* we mean $S \setminus \{0\}$.

We begin by recalling that an atomic domain is a domain where each nonzero nonunit can be written as a (finite) product of irreducibles (or atoms).

An atomic domain R is called a half factorial domain (HFD) if, given any collection $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n\}$ of irreducible elements of R with

$$\alpha_1\alpha_2\cdots\alpha_m = \beta_1\beta_2\cdots\beta_n,$$

we have $n = m$.

Zaks introduced the term "half factorial domain" in [8], although they were first studied by Carlitz in [2]. In [6], the author used the following tool to study overrings of a general HFD.

E-mail address: jack.l.maney@ndsu.nodak.edu.

0021-8693/$ – see front matter
doi:10.1016/S0021-8693(03)00433-2

Definition 1.1. Let R be an HFD with quotient field K. If $R \neq K$, we define the boundary map $\partial_R : K^* \to \mathbb{Z}$ by $\partial_R(\alpha) = t - s$ where

$$\alpha = \frac{\pi_1 \pi_2 \cdots \pi_t}{\delta_1 \delta_2 \cdots \delta_s}$$

and where π_i, δ_j are irreducibles of R for all i and j. If $R = K$, we say $\partial_R(\alpha) = 0$ for all nonzero α.

It is easy to see that ∂_R is a homomorphism of abelian groups. Also, given any $x \in R^*$, $\partial_R(x)$ counts out the number of irreducibles in any irreducible factorization of x, and for all $x \in R^*$, $\partial_R(x) = 0$ if and only if $x \in U(R)$.

Two classes of overrings defined with respect to the boundary map were studied in [6]. They are redefined here for the sake of completeness.

Definition 1.2. Let R be an HFD with quotient field K, and let T be an overring of R. We call T a boundary positive overring of R if for all nonzero $x \in T$, $\partial_R(x) \geqslant 0$.

Other equivalent terminology to that expressed in Definition 1.2 would be that $R \subseteq T$ is a boundary positive extension or T is boundary positive over R.

Definition 1.3. Let R be an HFD with quotient field K, and let T be an overring of R. We say that T is a boundary complete overring with respect to R if $x \in T$ with $\partial_R(x) = 0$ implies $x \in U(T)$.

In other words, T is boundary complete over R if no nonunit of T has zero boundary (over R, of course). Other equivalent terminology to that expressed in Definition 1.3 would be that $R \subseteq T$ is boundary complete, or T is boundary complete over R.

In [6], we found the following example of an HFD that had a boundary positive overring that was not boundary complete. This example is an example of a domain that is "almost" a valuation domain. We explore the properties of these HFDs in the next section.

Example 1.4. Let F be any field, and let $K = F(x)$. Set $R = F + tK[\![t]\!]$. It is easy to see that R is an HFD, and, in fact, any irreducible of R is of the form ut for $u \in U(K[\![t]\!])$. Consider the overring $T = F[x] + tK[\![t]\!]$. It is an easy check to see that T is boundary positive over R, since $T \subseteq K[\![t]\!]$ and $K[\![t]\!]$ is a boundary positive overring of R. However, T is not boundary complete, since $x \in T^* \setminus U(T)$ and $\partial_R(x) = \partial_R(xt/t) = 1 - 1 = 0$.

Also, given any α in the quotient field of R, we may write $\alpha = ut^n$ where $u \in U(K[\![t]\!])$ and $n = \partial_R(\alpha)$. It is clear that if $\partial_R(\alpha) > 0$ then $\alpha \in R$, and if $\partial_R(\alpha) < 0$, then $\alpha^{-1} \in R$.

2. Boundary valuation domains

Example 1.4 motivates the following definition.

Definition 2.1. Let R be an HFD with quotient field K. We say that R is a boundary valuation domain (BVD) if for every $\alpha \in K^*$ with $\partial_R(\alpha) \neq 0$, either α or α^{-1} is in R.

We give here an example of an HFD that is not a BVD.

Example 2.2. Let $R = \mathbb{Z}$. Then it is clear that R is an HFD. However, R is not a BVD, for given primes p, q, and r with $r \nmid pq$, we see that $\partial_R(pq/r) \neq 0$, but $pq/r, r/(pq) \notin R$.

In this section, R is an HFD with quotient field K. Also, R' will denote the complete integral closure of R.

Theorem 2.3. *The following three conditions are equivalent*:

(1) *R is a BVD.*
(2) *For $x, y \in R^*$ with $\partial_R(x) < \partial_R(y)$ or $0 = \partial_R(x) = \partial_R(y)$, we have $x \mid y$ in R.*
(3) *If $x \in K^*$ with $\partial_R(x) > 0$, then $x \in R$.*

Furthermore, conditions (1)–(3) *imply the following two conditions*:

(4) *Given any boundary positive extension $R \subsetneq T$ and $x \in T \setminus R$, we have $\partial_R(x) = 0$.*
(5) *Given any boundary complete extension $R \subsetneq T$, we have $T \setminus R \subseteq U(T)$.*

Proof. (1) $\Rightarrow$ (2). Choose $x, y \in R$. If $\partial_R(x) = 0$, then $x \mid y$, since $x \in U(R)$. So, assume $0 < \partial_R(x) = s < \partial_R(y) = t$. Write $x = \delta_1\delta_2\cdots\delta_s$ and $y = \pi_1\pi_2\cdots\pi_t$ with δ_i, π_j irreducibles in R, and $t > s$. Consider $y/x = \pi_1\cdots\pi_t/(\delta_1\cdots\delta_s)$. Note that $\partial_R(y/x) = t - s > 0$. If $y/x \notin R$, then, since R is a boundary valuation domain, we must have $x/y \in R$. However, $\partial_R(x/y) < 0$, which is a contradiction, since no element of R can have negative boundary over R. Therefore $y/x \in R$, $xy/x = y$, and $x \mid y$ in R.

(2) $\Rightarrow$ (3). Let $x \in K^*$ with $\partial_R(x) > 0$. Write $x = \pi_1\cdots\pi_t/(\delta_1\cdots\delta_s)$ with, as usual, π_i, δ_j irreducibles, and $t > s$. Let $a = \pi_1\cdots\pi_t$, $b = \delta_1\cdots\delta_s$. Then, $a, b \in R$ with $0 < \partial_R(b) < \partial_R(a)$. So, by 2, $b \mid a$ in R. That is, $\exists r \in R$ such that $br = a$, whence $r = a/b = x \in R$.

(3) $\Rightarrow$ (1). Let $x \in K^*$ with $\partial_R(x) \neq 0$. If $\partial_R(x) > 0$, then $x \in R$, by hypothesis. If $\partial_R(x) < 0$, then $\partial_R(x^{-1}) > 0$ implies that $x^{-1} \in R$, again, by hypothesis. Therefore R is a BVD.

(1)–(3) $\Rightarrow$ (4), (5). The proof of this assertion is obvious, and is left to the reader. □

Corollary 2.4. *Let R be a BVD with boundary complete overring T $(T \neq K)$. Then T is quasi-local and* $\dim(T) = 1$. *In particular, R is quasi-local and* $\dim(R) = 1$.

Proof. Let T be a boundary complete overring of R with $T \neq K$. Note that T is also boundary positive by [6, Theorem 3.2], and T is atomic by [6, Proposition 3.11].

It suffices to show that given any nonzero prime ideal $\mathcal{P}$ of T, $\mathcal{P}$ contains all irreducible elements of T.

So, let $\mathcal{P}$ be any nonzero prime ideal of T. Then $\mathcal{P}$ must contain an irreducible element of T, call it x. Let y be any other irreducible element of T. Clearly, $\partial_R(x), \partial_R(y) > 0$, and by Theorem 2.3, $x, y \in R$.

So, choose n such that $\partial_R(x) < \partial_R(y^n)$. Then, by Theorem 2.3, $x \mid y^n$ in R. Thus, there exists $r \in R$ such that $xr = y^n$, which implies that $y^n \in \mathcal{P}$, whence $y \in \mathcal{P}$. Since y was an arbitrary irreducible element of T, $\mathcal{P}$ contains all irreducibles of T, whence $\mathcal{P}$ is the unique nonzero prime ideal of T. □

Before we continue, we present a result from [6] that we will need for the next theorem.

Proposition 2.5. *Let R be an HFD. Then any almost integral extension of R is boundary positive.*

Proof. Let T be any almost integral overring of R, and let $x \in T$. Then there exists $r \in R^*$ such that $rx^n \in R$ for all $n \geqslant 0$. Thus

$$\partial_R(rx^n) = \partial_R(r) + n\partial_R(x) \geqslant 0,$$

and, since this works for all $n \geqslant 0$, we see that $\partial_R(x) \geqslant 0$. □

Theorem 2.6. *Let R be a BVD with quotient field K and complete integral closure R'. Let T be an overring of R. Then*:

(1) *R' is the unique maximal boundary positive overring of R, and is boundary complete.*
(2) *R' is a Rank 1 DVR.*
(3) *If T is boundary positive, but not boundary complete, then T is not atomic.*
(4) *If T is an HFD, then T is a BVD.*
(5) *If T is boundary complete over R, then T is a BVD.*

Proof. For (1), suppose A is a boundary positive overring of R, and choose $x \in A$. If $\partial_R(x) > 0$, then $x \in R$ and $x \in R'$ all the more so. If $\partial_R(x) = 0$, then, by Theorem 2.3, given any $y \in R$ with $\partial_R(y) > 0$, we have, for all $n \geqslant 0$, $\partial_R(yx^n) = \partial_R(y) > 0$, whence $yx^n \in R$. Thus $x \in R'$, and $A \subseteq R'$. Thus any boundary positive overring is contained inside of R', whence R' is the unique maximal boundary positive overring of R.

The fact that R' is the unique maximal boundary positive overring of R implies that R' is also boundary complete. For if R' is not boundary complete, consider the overring R'_S of R', where $S = \{x \in R' \mid \partial_R(x) = 0\}$. It is clear that R'_S is boundary complete and boundary positive, and that $R' \subseteq R'_S$. But then we must have $R' = R'_S$, by maximality of R', whence R' is boundary complete.

For (2), let $\alpha \in K^*$. If $\partial_R(\alpha) > 0$, then $\alpha \in R \Rightarrow \alpha \in R'$. If $\partial_R(\alpha) < 0$, then $\alpha^{-1} \in R \subseteq R'$. If $\partial_R(\alpha) = 0$, then given any $y \in R^*$ with $\partial_R(y) > 0$, we have, for all $n \in \mathbb{N}$, $\partial_R(y\alpha^n) = \partial_R(y) > 0$ which implies that $y\alpha^n \in R$. Thus $\alpha \in R'$. So, since R' is boundary positive and boundary complete, it must be atomic [6, Proposition 3.11]). Therefore R' is a Rank 1 DVR.

For (3), assume that T is boundary positive and not boundary complete. Let x be a nonzero nonunit of T with $\partial_R(x) = 0$, and let $y \in R$ be irreducible. Since T is boundary positive, y must be a nonunit of T. If T is atomic, then we may write

$$y = \pi_1 \cdots \pi_k$$

where each π_i is irreducible in T. Since $\partial_R(y) = 1$ and $\partial_R(\pi_i) \geqslant 0$ for each i, we may say—without loss of generality—that $\partial_R(\pi_1) = 1$ and $\partial_R(\pi_i) = 0$ for $2 \leqslant i \leqslant n$. Then, π_1/x is a nonunit of R (since $\partial_R(\pi_1/x) = 1$), whence it is a nonunit of T. This, however, gives us that $\pi_1 = x(\pi_1/x)$, and we have written π_1 as a product of two nonunits in T, a contradiction. Thus T is not atomic.

For (4), let $\alpha \in K^*$, with $\partial_R(\alpha) \neq 0$. Since R is a boundary valuation domain, either α or α^{-1} is an element of R. Thus, either α or α^{-1} is an element of T, and since T is an HFD, this implies that T is a boundary valuation domain.

For (5), the result holds vacuously if $T = K$, so we may (by [6, Theorem 3.2]) assume that $T \neq K$, whence T is boundary positive and boundary complete over R. We know, by [6, Proposition 3.11] that T is atomic. So, let x be any irreducible element of T. We must have $\partial_R(x) > 0$. If $\partial_R(x) \geqslant 2$, then $\partial_R(x/\pi) \geqslant 1$, where π is an irreducible of R. But then $x = x\pi/\pi$ and we have written x as a product of two nonunits of T, a contradiction. Thus $\partial_R(x) = 1$, and by [6, Lemma 3.12], T is an HFD, whence T is a BVD by (4). □

We now give an example of an HFD that is not a BVD, and whose complete integral closure is a Rank 1 DVR. Before we do this, we will use the following lemma that is part of Lemma 6.6 in [1].

Lemma 2.7 [1]. *Let $F \subseteq K$ be an extension of fields. Let $V \subseteq K$ be a vector space over F with $V^2 := \{uv \mid u, v \in V\} = K$. Then $R = F + xV + x^2K[\![x]\!]$ is an HFD.*

Proof. It suffices to show that every irreducible element has "least degree" 1—i.e. every irreducible element of R is of the form $xg(x)$ where $g(0) \in V^*$. Let $f(x) = a_2x^2 + a_3x^3 + \cdots \in R$. Since $a_2 \in K = V^2$, we may write $a_2 = \alpha\beta$ for some $\alpha, \beta \in V$. Thus $f(x) = (\alpha x)(\beta x + a_3x^2 + \cdots)$ is not irreducible. □

Example 2.8 [1, *Example* 6.7]. Let y be a root of the polynomial $X^4 + X + 1 \in \mathbb{F}_2[X]$, and let $\{1, y, y^2, y^3\}$ be a basis of $K = \mathbb{F}_{16}$ over $F = \mathbb{F}_2$. Let V be the vector space over F with basis $\{1, y, y^2\}$. Let $R = F + xV + x^2K[\![x]\!]$. It is an easy check to see that $V^2 = K$. Thus R is an HFD by Lemma 2.7.

Also, it is an easy check to see that the complete integral closure of R is $R' = K[\![x]\!]$, which is a Rank 1 DVR.

However, $\partial_R(x/y) = 1 > 0$, but if $x/y \in R$, then we must have $1/y \in V$. But if we write $1/y = \alpha_0 + \alpha_1 y + \alpha_2 y^2$ where $\alpha_i \in F$, then $1 = \alpha_0 y + \alpha_1 y^2 + \alpha_2 y^3$, violating the linear independence of $\{1, y, y^2, y^3\}$. Thus $x/y \notin R$, and R is not a BVD.

In [3], the author asks if T is an atomic overring of an HFD R, then is T boundary complete? Theorem 2.6 allows us to answer this in the affirmative when R is a BVD.

Corollary 2.9. *Let R be a BVD with quotient field K and overring T. If T is atomic, then T is boundary complete. Moreover, if T is not boundary positive, then $T = K$.*

Proof. There are two cases.

First, suppose T is boundary positive. If T is not boundary complete, then by Theorem 2.6, T is not atomic, a contradiction.

So, suppose T is not boundary positive. Then, $R' \subsetneq T \subseteq K$. But R' is a Rank-1 DVR, by Theorem 2.6. However, the only overring that strictly contains a Rank-1 DVR is its quotient field, whence $T = K$. $\square$

Corollary 2.10. *Let R be a noetherian boundary valuation domain. Then every boundary positive overring of R is a BVD.*

Proof. Let T be a boundary positive overring of R. Since R is one-dimensional and noetherian, every overring of R must also be noetherian and of dimension at most 1 [5, Theorem 93]. Thus T is atomic, whence T is boundary complete. Hence, T is a BVD by Theorem 2.6. $\square$

Corollary 2.11. *Let R be a BVD with a boundary positive overring T. Then the following are equivalent*:

(1) *T is boundary complete over R.*
(2) *T is a BVD.*
(3) *T is quasi-local and* $\dim(T) = 1$.

Proof. $(1) \Rightarrow (2)$. This is merely a restatement of Theorem 2.6(4).

$(2) \Rightarrow (3)$. Since T is atomic, T must be boundary complete by Theorem 2.6. Thus, by Corollary 2.4, T is quasi-local and one-dimensional.

$(3) \Rightarrow (1)$. Let M be the unique nonzero prime ideal of T. Consider $S = \{x \in T^* \mid \partial_R(x) = 0\}$. It is easy to see that S is a multiplicative subset of T. Since T is boundary positive, it is also easy to see that S is saturated. Thus $S = T \setminus M = U(T)$, and T is boundary complete. Therefore, by Theorem 2.6, T is a BVD. $\square$

Theorem 2.12. *The following conditions on a BVD R are equivalent*:

(1) *R is completely integrally closed.*
(2) *There exists a nonzero prime element of R.*
(3) *R is a Dedekind domain.*
(4) *R is a Rank* 1 *DVR.*

Proof. $(1) \Rightarrow (2)$. Since $R = R'$ is a Rank 1 DVR (which contains a nonzero prime), it follows that R contains a nonzero prime element.

$(2) \Rightarrow (3)$. Let $x \in R^*$ be a prime element of R.

Let $y \in R$ be any irreducible element. Then $\partial_R(x) = 1 < 2 = \partial_R(y^2)$ which, since R is a boundary valuation domain, implies that $x \mid y^2$ in R. But x is prime, so $x \mid y$. Since x

and y are both irreducible elements, and one divides the other, we conclude that x and y are associates. Since any associate of a prime element is again prime, we conclude that y is prime. Therefore R is a PID, and hence a Dedekind domain.

(3) $\Rightarrow$ (4). Suppose R is a Dedekind domain. Since R is one-dimensional and quasi-local, it follows that R is a Rank 1 DVR.

(4) $\Rightarrow$ (1). This assertion clearly follows. $\square$

3. The boundary map and the group of divisibility

Recall that, given a domain R with quotient field K, then the group of divisibility of R is the abelian group $G = G(R) = U(K)/U(R) = (K^*)/(U(R))$. This is a partially ordered group under the operation $\leqslant$, where $aU(R) \leqslant bU(R)$ if and only if $a \mid b$ *in* R, or equivalently, $b/a \in R$. Also, we denote the positive elements of $G(R)$ by $G^+ = G(R)^+ = \{aU(R) \in G(R) \mid aU(R) \geqslant U(R)\}$. Note that the coset representatives of G^+ are precisely the nonzero elements of R.

It is easy to prove that the group of divisibility of any Rank-1 DVR is $\mathbb{Z}$ under the usual order. For excellent discussions of groups of divisibility, see [4,7].

The following theorem essentially says that if R is an HFD, then the boundary map can be extended to the group of divisibility in a very natural way.

Theorem 3.1. *Let R be an HFD with quotient field K, and let $G = G(R)$ be the group of divisibility of R. Then the map*

$$\partial_R : \big(U(K)/U(R)\big) = G \longrightarrow \mathbb{Z}$$

given by $\partial_R(\alpha U(R)) = \partial_R(\alpha)$ is a well-defined homomorphism of abelian groups.

Proof. Let $\alpha U(R) = \beta U(R)$ in G. Then there exists some $u \in U(R)$ such that $\alpha = u\beta$. So,

$$\partial_R\big(\alpha U(R)\big) = \partial_R(\alpha) = \partial_R(u\beta) = \partial_R(\beta) = \partial_R\big(\beta U(R)\big)$$

and ∂_R is well-defined.

The fact that ∂_R is a homomorphism on G follows from the fact that ∂_R is a homomorphism on K^* and the fact that, in G, we have the operation $(\alpha U(R))(\beta U(R)) = \alpha\beta U(R)$. $\square$

Proposition 3.2. *Let R be an HFD with quotient field K, and group of divisibility $G(R)$. Then $\alpha U(R) \leqslant \beta U(R)$ implies that $\partial_R(\alpha U(R)) \leqslant \partial_R(\beta U(R))$.*

Proof. Write $\alpha = a/b$, $\beta = c/d$, with $a, b, c, d \in R^*$. By definition of the partial ordering on $G(R)$, $\beta/\alpha = cb/(ad) \in R$. So, $\partial_R(cb/(ad)) \geqslant 0$, whence $\partial_R(c) + \partial_R(b) - \partial_R(a) - \partial_R(d) \geqslant 0$. Therefore $\partial_R(c/d) = \partial_R(\beta) \geqslant \partial_R(a/b) = \partial_R(\alpha)$. $\square$

We will characterize BVD's via their groups of divisibility. Our process requires several steps.

Theorem 3.3. *Let R be a BVD with quotient field K and complete integral closure R'. Then $G(R) \cong \mathbb{Z} \oplus (U(R')/U(R))$ with the following order*:

$$\begin{aligned}\bigl(n, uU(R)\bigr) \leqslant \bigl(m, vU(R)\bigr) \quad \textit{if and only if} \quad & n < m \quad \textit{or} \\ & n = m \quad \textit{and} \quad vu^{-1} \in U(R). \qquad (1)\end{aligned}$$

Proof. Note that in $\mathbb{Z} \oplus (U(R')/U(R))$, we denote the operation by addition, but within 2-vectors, we write the second coordinate multiplicatively as $(n_1, uU(R)) + (n_2, vU(R)) = (n_1 + n_2, uvU(R))$.

Recall that R' is a Rank-1 DVR. So, let z be the unique prime of R' (up to a unit in R'). Any nonunit of R is of the form uz^n for $n > 0$ and $u \in U(R')$. Therefore, any element of K^* is of the form uz^n for $z \in \mathbb{Z}$ and $u \in U(R')$.

Consider the map $\phi : G(R) \to \mathbb{Z} \oplus (U(R')/U(R))$ given by $\phi(uz^n U(R)) = (n, uU(R))$. It is clear that this is a well defined function.

To show that ϕ is a homomorphism, simply observe that if $z^n uU(R), z^m vU(R) \in G(R)$, then

$$\begin{aligned}\phi\bigl(\bigl(z^n uU(R)\bigr)\bigl(z^m vU(R)\bigr)\bigr) &= \phi\bigl(z^{n+m} uvU(R)\bigr) = \bigl(n + m, uvU(R)\bigr) \\ &= \bigl(n, uU(R)\bigr) + \bigl(m, vU(R)\bigr) = \phi\bigl(z^n uU(R)\bigr) + \phi\bigl(z^m vU(R)\bigr),\end{aligned}$$

whence ϕ is a homomorphism.

ϕ is clearly onto, for given $(n, uU(R)) \in \mathbb{Z} \oplus (U(R')/U(R))$, $\phi(z^n uU(R)) = (n, uU(R))$.

If $z^n uU(R) \in \operatorname{Ker}(\phi)$, then $z^n uU(R) \mapsto (0, U(R))$, implying that $n = 0$ and $u \in U(R)$, whence $z^n uU(R) = U(R)$. Therefore $\operatorname{Ker}(\phi)$ is trivial, and ϕ is one to one.

The verification of the ordering on $G(R)$ is left to the reader. □

Now we work our way to the converse—that is, if R is a domain with quotient field K and complete integral closure R', and if $G = G(R) = \mathbb{Z} \oplus (U(R')/U(R))$ (with the partial ordering (1)), then R is a BVD.

In G, we will denote the element $(n, uU(R))$ by (n, u), suppressing the implicit fact that u is a coset representative of a coset of $U(R')/U(R)$. Also, as in the proof of Theorem 3.3, we will denote the addition of two elements of G by addition in the first coordinate and multiplication in the second coordinate.

Before continuing, we recall that an element of R is irreducible if and only if its coset representative is a minimal (strictly) positive element in G.

Lemma 3.4. *Let R be a domain with quotient field K, complete integral closure R', and $G(R) = \mathbb{Z} \oplus U(R')/U(R)$ under the partial order* (1). *Then, all minimal positive elements in $G(R)$ are of the form $(1, u)$ for $uU(R) \in U(R')/U(R)$, and all elements in $G(R)$ of the* *form* $(0, u)$ *correspond to units of* R'*. Consequently, all irreducibles of* R *correspond to elements of the form* $(1, u) \in G(R)$.

Proof. Fix $y = (1, u)$ for some $u \in U(R')$, and let $x = (m, v)$ with $(0, 1) \leqslant (m, v) \leqslant (1, u)$ in G. Note, of course, that $(0, 1)$ is the identity element of G.

There are two cases.

If $m = 0$, then $(0, 1) \leqslant (m, v)$ which implies that $v \in U(R)$, whence $x = (m, v) = (0, 1)$, and x is not strictly positive.

If $m = 1$, then $(1, v) \leqslant (1, u)$ implies $uv^{-1} \in U(R)$, whence $vu^{-1} \in U(R)$. Therefore $(1, u) \leqslant (1, v)$, and we see that $(1, u) = (1, v)$. Thus $x = y$, and $(1, u)$ is minimal positive.

On the other hand, let (n, u) be minimal positive. Then we must have $n = 1$; otherwise $(0, u) < (n - 1, u) < (n, u)$, violating the minimality of (n, u). Thus $(n, u) = (1, u)$ is of the required form.

Now, consider an element of the form $(0, u)$. If we choose a positive element of the form $(1, v)$, we see that for each $n \in \mathbb{N}$,

$$(1, v) + n(0, u) = \bigl(1, vu^n\bigr) > (0, 1).$$

Thus given the element α of K^* corresponding (up to a unit in R) to $(0, u)$, we see that we can find some $r \in R$ such that $r\alpha^n \in R$ for all $n \geqslant 0$. Thus $\alpha \in R'$, and by the exact same argument, $\alpha^{-1} \in R'$. Thus $\alpha \in U(R')$.

Now, choose $\alpha \in U(R')$, and let (n, u) correspond to α in G. Suppose $n > 0$. Then $\alpha^{-1} \in R'$ and α^{-1} corresponds to $(-n, u^{-1}) \in G$. There exists some $r \in R^*$ such that for any $k \geqslant 0$, $r\alpha^{-k} \in R$. Let r correspond to (m, v). Pick k such that $m - kn < 0$. Then $r\alpha^{-k} \in R$, but in G, $r\alpha^{-k}$ corresponds to $(m - kn, vu^{-k}) \in G^+$. This is a contradiction. Thus we must have $n \leqslant 0$, and a symmetric argument gives us that we must have $n \geqslant 0$. Therefore α must correspond to an element of the form $(0, u)$. □

Lemma 3.5. *Let* R *be as in Lemma* 3.4*. Then* R *is an HFD.*

Proof. The idea of the first part of the proof is nearly identical to that of the proof of Proposition 3.11, part (i) of [6].

Let y be any nonzero nonunit element of R, and let $y = (n, u)$ in G with $n > 1$. Suppose y cannot be written as a product of irreducibles in R. Then, we may write $y = \alpha_1\beta_1$ with α_1, β_1 nonunits of R, and (without loss of generality) β_1 cannot be written as a product of irreducibles of R. Let $\alpha_1 = (m_1, u_1)$, $\beta_1 = (n_1, v_1)$ in G $(m_1, n_1 > 0)$.

Since β_1 cannot be factored into irreducibles, we may likewise write $\beta_1 = \alpha_2\beta_2$ with $\alpha_2, \beta_2 \in R \setminus U(R)$ and (without loss of generality) β_2 cannot be written as a product of irreducibles of R. Let $\alpha_2 = (m_2, u_2)$, $\beta_2 = (n_2, v_2)$ in G $(m_2, n_2 > 0)$.

Continue this process. At the kth step for $k \geqslant n$, we have

$$y = \alpha_1 \cdots \alpha_k \beta_k$$

which gives, in G,

$$(n,u)=(n_k,v_k)+\sum_{i=1}^{k}(m_i,u_i)=\left(n_k+\sum_{i=1}^{k}m_i,\ v_k\prod_{i=1}^{k}u_i\right).$$

This is a contradiction, since $n_k+\sum_{i=1}^{k}m_i>n$. Therefore R is atomic.

To see why R is an HFD, let

$$\alpha_1\alpha_2\cdots\alpha_m=\beta_1\beta_2\cdots\beta_n$$

be two irreducible factorizations in R. Since each α_i,β_j are irreducible, we see, by Lemma 3.4, that α_i and β_j are of the form $(1,u_i)$ and $(1,v_j)$, respectively, in G. So, looking at the images of these irreducibles in G, and combining, we see that

$$\left(\sum_{i=1}^{m}1,\ \prod_{i=1}^{m}u_i\right)=\left(\sum_{j=1}^{n}1,\ \prod_{j=1}^{n}v_j\right),$$

which implies that $n=m$. $\square$

Lemma 3.6. *Let R be as in Lemma* 3.4. *Then given any element $(n,u)\in G$, $\partial_R((n,u))=n$.*

Proof. If $n=0$, then $(n,u)=(0,u)$ corresponds to some unit in R', by Lemma 3.4. Since R' is a boundary positive extension of R (by [6, Proposition 3.10]), any unit of R' has boundary zero. Therefore $\partial_R((0,u))=0$.

If $n=1$, then $(1,u)$ corresponds to an irreducible element of R, whence $\partial_R((1,u))=1$.

Suppose $n>1$. Then, by Lemma 3.5, any element corresponding to (n,u) in G is an element of R factoring into exactly n irreducibles. Therefore $\partial_R((n,u))=n$.

So, suppose $n<0$. Let $\alpha\in K\setminus\{0\}$ with $\alpha=(n,u)$ in G. Then $\alpha^{-1}=(-n,u)$, and by the above, $\partial_R(\alpha^{-1})=\partial_R((-n,u))=-n$. Therefore, since $\alpha\alpha^{-1}=1$, we see that

$$\partial_R(\alpha)+\partial_R\big(\alpha^{-1}\big)=\partial_R(1)=0\quad\Rightarrow\quad\partial_R(\alpha)-n=0$$

whence $\partial_R(\alpha)=\partial_R((n,u))=n$. $\square$

We now characterize BVD's by their groups of divisibility.

Theorem 3.7. *Let R be a domain with quotient field K and complete integral closure R', and let $G=G(R)$ be the group of divisibility of R. Then R is a BVD if and only if*

$$G\cong\mathbb{Z}\oplus\big(U(R')/U(R)\big)$$

with the partial order (1).

Proof. By Theorem 3.3, the condition on G is necessary. We now show that it is sufficient.

By Lemma 3.5, R is an HFD, so let $\alpha \in K^*$ be such that $\partial_R(\alpha) = n > 0$. Then, in G, α is of the form (n, u) for some $u \in U(R')$. However, this implies that $(0, 1) \leqslant (n, u)$ and (n, u) is a positive element of G, which is to say $\alpha \in R$. So, by Theorem 2.3, R is a BVD. $\square$

Acknowledgments

The author thanks the NDSU Mathematics Department, and especially his graduate advisor, Jim Coykendall. The author also thanks the referee for the numerous small changes that, together, greatly improved the presentation of this paper.

References

[1] D.D. Anderson, J.L. Mott, Cohen–Kaplansky domains: integral domains with a finite number of irreducible elements, J. Algebra 148 (1992) 17–41.
[2] L. Carlitz, A characterization of algebraic number fields with class number two, Proc. Amer. Math. Soc. 11 (1960) 391–392.
[3] S. Chapman, J. Coykendall, Half-factorial domains, a survey, in: Non-Noetherian Commutative Ring Theory, in: Math. Appl, Kluwer Academic, Boston, 2000, pp. 97–115.
[4] R. Gilmer, Multiplicative Ideal Theory, in: Queen's Papers in Pure and Appl. Math., vol. 90, Queen's Univ., Kingston, Ontario, Canada, 1992.
[5] I. Kaplansky, Commutative Rings, Polygonal Publishing House, Washington, NJ, 1974.
[6] J. Maney, On the boundary map and overrings, submitted for publication.
[7] J. Mott, Groups of divisibility: A unifying concept for integral domains and partially ordered groups, Math. Appl. 48 (1989) 80–104.
[8] A. Zaks, Half factorial domains, Bull. Amer. Math. Soc. 82 (1976) 721–723.